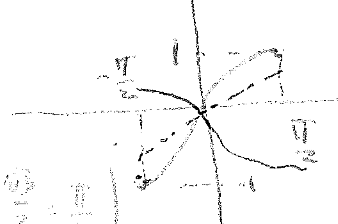
Calculus II (Lesson #30) March 1, 2013

We restrict the trig functions to "preferred" domains to get bijective functions:

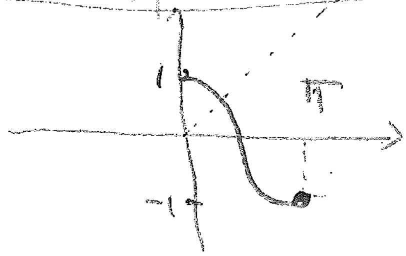
$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$



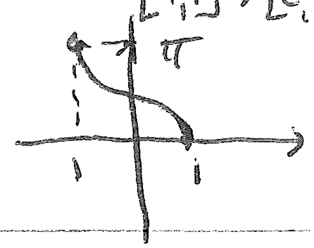
$\arcsin : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\arcsin \frac{1}{2} = \frac{\pi}{6}$ $\quad \arcsin \frac{\sqrt{2}}{2} = \frac{\pi}{4}$ $\quad \arcsin \frac{\sqrt{3}}{2} = \frac{\pi}{3}$

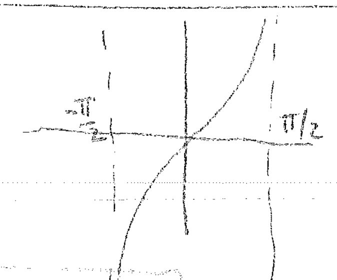
$\cos : [0, \pi] \to [-1, 1]$

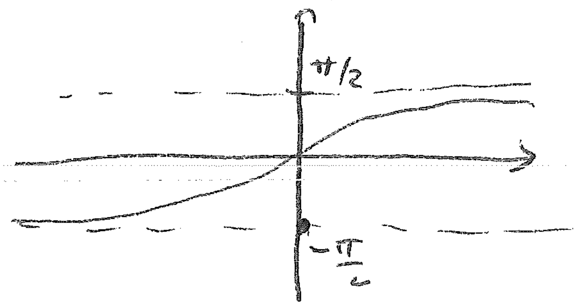


$\arccos : [-1,1] \to [0, \pi]$



$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$





$\arctan \frac{\sqrt{3}}{3} = \frac{\pi}{6}$ $\quad \arctan 1 = \frac{\pi}{4}$ $\quad \arctan \sqrt{3} = \frac{\pi}{3}$

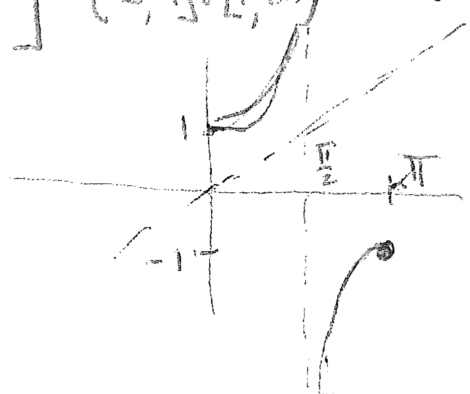
$\sec : \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \to (-\infty, -1] \cup [1, \infty)$

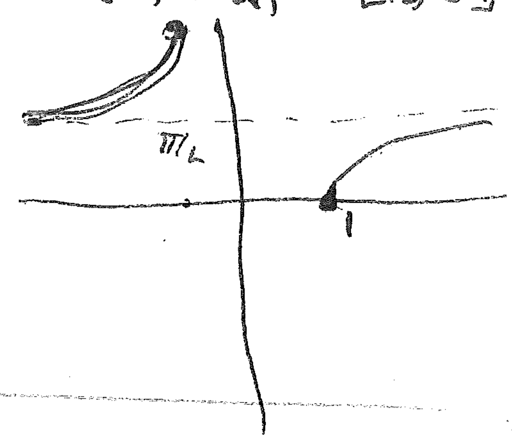
$\text{arcsec} : (-\infty, -1] \cup [1, \infty) \to \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

$\text{arcsec} \frac{2}{\sqrt{3}} = \frac{\pi}{6}$

$\text{arcsec} \sqrt{2} = \frac{\pi}{4}$

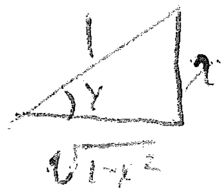
$\text{arcsec}\, 2 = \frac{\pi}{3}$





We use "representative triangles"
to find derivatives of inverse trig functions

Let $y = \arcsin x$



Then $\sin y = x$

so $\cos y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \sec y = \boxed{\frac{1}{\sqrt{1-x^2}}}$

$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$

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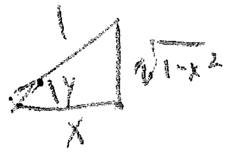
let $y = \arccos x$

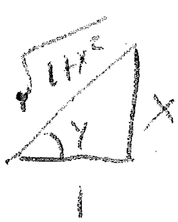
Then $\cos y = x \Rightarrow -\sin y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = -\csc y$



$\boxed{\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}}$

Let $y = \arctan x$

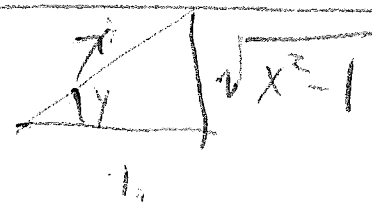
So $\tan y = x \Rightarrow \sec^2 y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \cos^2 y$



$\cos y = \frac{1}{\sqrt{1+x^2}}$

$\Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{1+x^2}}$

Let $y = \operatorname{arcsec} x$



So $\sec y = x$

So $\sec y \tan y \frac{dy}{dx} = 1$

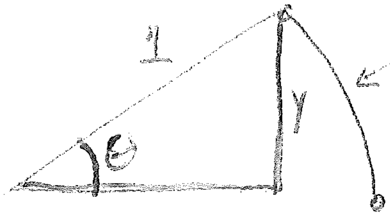
So $\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{x\sqrt{x^2-1}}$

$\sec y = x$

$\tan y = \sqrt{x^2-1}$

~~If $x<0$, tan y works out to~~

The "arc" in arcsin



length of arc on unit circle
is the angle θ

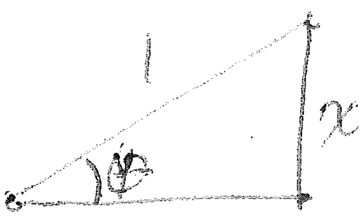
$\arcsin y = \theta \Leftrightarrow \sin\theta = y$

arcsin y is an angle
arctan y is an angle

"Representative Triangles"

~~$\arcsin y = \theta$~~

~~$\arcsin y = \theta$~~



$\sqrt{1-x^2}$

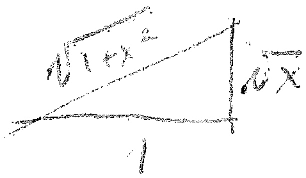
$\sin\theta = x$
$\cos\theta = \sqrt{1-x^2}$

~~$\theta = \arcsin y$~~

~~$\sin y = \theta$~~

~~$\cos y \frac{dy}{dx} = 1$~~

~~$\frac{dy}{dx} = \sec y$~~



$\tan\theta = \sqrt{x}$

$\sin\theta = \frac{\sqrt{x}}{\sqrt{1+x^2}}$

$\sec\theta = \sqrt{1+x^2}$

~~$y = \arcsin x$~~

~~$\frac{dy}{dx}$~~

~~$\sin y = x$~~

~~$\cos y \frac{dy}{dx} = 1$~~

~~$\frac{dy}{dx} = \sec y$~~

These derivatives tell us:

$\rightarrow \quad \displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \operatorname{arcsec} x + C$

$\rightarrow \quad \displaystyle\int \frac{1}{1+x^2}\,dx = \arctan x + C$

$\rightarrow \quad \displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx = \operatorname{arcsec} x + C$

Example $\displaystyle\int_1^{\sqrt{2}} \frac{dx}{\sqrt{4-x^2}}$

$$= \int_1^{\sqrt{2}} \frac{dx}{2\sqrt{1-\left(\frac{x}{2}\right)^2}}$$

let $u = \frac{x}{2}$

so $du = \frac{1}{2}dx$

and $x = \sqrt{2} \Rightarrow u = \frac{\sqrt{2}}{2}$

and $x = 1 \Rightarrow u = \frac{1}{2}$

$$= \int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \frac{du}{\sqrt{1-u^2}}$$

$$= \arcsin u \Big|_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} = \arcsin\frac{\sqrt{2}}{2} - \arcsin\frac{1}{2}$$

$$= \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$

Complete the square

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x^2+2x+1)+1}$$

$$= \int \frac{dx}{(x+1)^2+1} \qquad \text{Let } u = x+1, \quad du = dx$$

$$= \int \frac{du}{u^2+1}$$

$$= \arctan u + C$$

$$= \arctan(x+1) + C$$

$$\int \frac{dx}{\sqrt{e^{2x}-1}}$$

$$= \int \frac{e^x\,dx}{e^x\sqrt{e^{2x}-1}} \qquad \text{Let } u = e^x, \quad du = e^x\,dx$$

$$= \int \frac{du}{u\sqrt{u^2-1}}$$

$$= \operatorname{arcsec} u + C$$

$$= \operatorname{arcsec}(e^x) + C$$